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On motions of test-particles in Kerr metric

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Motions of test-particles in the $\theta = \pi/2$ plane of the Kerr metric of a rotating body have been studied here under some specific conditions. Corrections have also been introduced for their spins.

Introduction

A particular solution of Einstein's field equations of great interest is Kerr solution (Kerr 1963) which is the only exact solution describing the field exterior to some finite rotating body. The Kerr metric can be written in Schwarzschild-like coordinates (Boyer & Lindquist 1967), *i.e.* coordinates such that when the angular momentum parameter is zero one gets the Schwarzschild metric. Thus we have

$$\begin{aligned} ds^2 = & -[(r^2+\alpha^2\cos^2\theta)/(r^2-2mr+\alpha^2)]dr^2-(r^2+\alpha^2\cos^2\theta)d\theta^2 \\ & -[r^2+\alpha^2+2mr\alpha^2\sin^2\theta/(r^2+\alpha^2\cos^2\theta)]\sin^2\theta\, d\phi^2 \\ & -[4m\alpha r\sin^2\theta/(r^2+\alpha^2\cos^2\theta)]dt\, d\phi+[1-2mr/(r^2+\alpha^2\cos^2\theta)]dt^2 \qquad \ldots (1) \end{aligned}$$

Boyer & Price (1965) have shown that "one of the parameters in Kerr's solution (equation 1) can plausibly be related to the angular momentum per unit mass of a uniformly rotating sphere, the other parameter being a measure of the mass of the sphere". Hernandez (1968) has also found a case such that in the Newtonian limit the rotating body in (1) is a uniform density sphere with a moment of inertia $= 2/5ma^2$ and angular momentum $= -m\alpha$.

We propose to study in this paper the motions of test-particles in the $\theta = \pi/2$ plane of the Kerr metric of a rotating body represented by (1) under some specific conditions. Corrections would also be introduced for their spins.

Orbital Equation in $\theta = \pi/2$ Plane

Upto second power in α and first power in m/r, the Kerr metric (1) is reduced to the form

$$\begin{aligned} ds^2 = & -\left(1+\frac{2m}{r}-\frac{\alpha^2}{r^2}\sin^2\theta\right)dr^2-(r^2+\alpha^2\cos^2\theta)d\theta^2-\sin^2\theta(r^2+\alpha^2)d\phi^2 \\ & -\frac{4m\,\alpha\sin^2\theta}{r}\,dt\,d\phi+\left(1-\frac{2m}{r}\right)dt^2 \qquad \ldots (2) \end{aligned}$$

The approximations made restricts the consideration to distances $r >> m$ and to cases where $\alpha << r$.

Corresponding to (2), the values of g^{11}, g^{22}, g^{33}, g^{34} and g^{44} are obtained. They are

$$
\begin{aligned}
g^{11} &= -\left(\frac{2m}{r}+\frac{\alpha^2}{r^2}\sin^2\theta\right)\\
g^{22} &= -\frac{1}{r^2}\left(1-\frac{\alpha^2}{r^2}\cos^2\theta\right)\\
g^{33} &= -\frac{1}{r^2\sin^2\theta}\left(1-\frac{\alpha^2}{r^2}\right)\\
g^{34} &= \frac{2m\alpha}{r^3}\\
g^{44} &= \left(1+\frac{2m}{r}\right)
\end{aligned} \qquad \ldots (3)
$$

Now, from (2) and (3), we can obtain the 3-index symbols and write out the θ-, ϕ- and t-equations. The θ-equation is

$$
\frac{d^2\theta}{ds^2}+\frac{\alpha^2\sin 2\theta}{2r^4}\left(\frac{dr}{ds}\right)^2+\frac{2}{r}\left(1-\frac{\alpha^2}{r^2}\cos^2\theta\right)\frac{dr}{ds}\frac{d\theta}{ds}-\tfrac{1}{2}\frac{\alpha^2}{r^2}\sin 2\theta\left(\frac{d\theta}{ds}\right)^2
$$

$$
-\tfrac{1}{2}\left(1+\frac{\alpha^2}{r^2}\sin^2\theta\right)\sin 2\theta\left(\frac{d\phi}{ds}\right)^2-\frac{2m\alpha}{r^3}\sin 2\theta\frac{d\phi}{ds}\frac{dt}{ds}=0. \qquad \ldots (4)
$$

Let us suppose that the test-particle moves initially in the $\theta = \pi/2$ plane. Then $d\theta/ds = 0$, and $\cos\theta = 0$ initially, so that $d^2\theta/ds^2 = 0$. The particle therefore continues to move in this plane, and we may simplify the ϕ- and t-equations by putting $\theta = \pi/2$ throughout.

Then, ϕ- and t-equations in $\theta = \pi/2$ plane are

$$
\frac{d^2\phi}{ds^2}+\frac{2}{r}\left(1-\frac{\alpha^2}{r^2}\right)\frac{dr}{ds}\frac{d\phi}{ds}-\frac{2m\alpha}{r^4}\frac{dr}{ds}\frac{dt}{ds}=0 \qquad (5)
$$

$$
\frac{d^2t}{ds^2}-\frac{2m\alpha}{r^2}\frac{dr}{ds}\frac{d\phi}{ds}+\left(1+\frac{2m}{r}\right)\left(\frac{2m}{r^2}\right)\frac{dr}{ds}\frac{dt}{ds}=0 \qquad (6)
$$

For $\alpha = 0$, the solutions of (5) and (6) are

$$
r^2\frac{d\phi}{ds}=h_0 \qquad (7)
$$

$$\left(1-\frac{2m}{r}\right)\frac{dt}{ds}=c_0 \qquad \ldots (8)$$

where h_0, c_0 are constants of integration. We now substitute from (7) and (8) in the correction terms of (5) and (6) and obtain the solutions

$$\frac{d\phi}{ds}=\frac{h}{r^2}\left(1-\frac{2m\alpha}{r}\frac{c_0}{h}-\frac{\alpha^2}{r^2}\frac{h_0}{h}\right) \qquad \ldots (9)$$

$$\frac{dt}{ds}=c\left(1+\frac{2m}{r}-\frac{2m\alpha}{3r^3}\frac{h_0}{c}\right) \qquad \ldots (10)$$

where h, c are constants of integration. Writing $h \approx h_0$ and $c \approx c_0$ in the correction terms in (9) and (10), we obtain finally

$$\frac{d\phi}{ds}=\frac{h}{r^2}\left(1-\frac{2m\alpha}{r}\frac{c}{h}-\frac{\alpha^2}{r^2}\right) \qquad \ldots (11)$$

$$\frac{dt}{ds}=c\left(1+\frac{2m}{r}-\frac{2m\alpha}{3r^3}\frac{h}{c}\right) \qquad \ldots (12)$$

Hence, in the $\theta=\pi/2$ plane, (2), (11) and (12) give in the usual manner the orbital equation for a test-particle

$$\frac{d^2u}{d\phi^2}+u=\frac{m}{h^2}\left\{1+2\alpha\frac{c}{h}(c^2-1)\right\}+\frac{3\alpha^2}{h^2}(c^2-1)u$$
$$+3m\left(1-\frac{8}{3}\alpha\frac{c}{h}\right)u^2-4\alpha^2u^3 \qquad \ldots (13)$$

where u has been substituted for $1/r$.

Advance of Perihelion

The solution of (13), neglecting all terms on the right side except the first, is

$$u=\frac{m}{h^2}\left\{1+2\alpha\frac{c}{h}(c^2-1)\right\}\{1+e\cos(\phi-\omega)\}. \qquad \ldots (14)$$

where e is the eccentricity of the orbit. Now, substituting from (14) on the right side of (13) and finding particular integrals for the terms containing $\cos(\phi-\omega)$, the final solution is

$$u=\frac{m}{h^2}\left\{1+2\alpha\frac{c}{h}(c^2-1)\right\}\{1+e\cos(\phi-\omega)\}+\frac{3m\alpha^2(c^2-1)e}{2h^4}\phi\sin(\phi-\omega)$$
$$+\frac{3m^3e}{h^4}\left\{1+4\alpha\frac{c}{h}(c^2-1)-\frac{8}{3}\alpha\frac{c}{h}\right\}\phi\sin(\phi-\omega)$$
$$-\frac{6m^3\alpha^2e}{h^6}\phi\sin(\phi-\omega)-\frac{3m^3\alpha^2e^3}{2h^6}\phi\sin(\phi-\omega). \qquad \ldots (15)$$

This gives for the advance of porihelion per revolution

$$\delta\omega = 2\pi\left[\frac{3}{2}\frac{\alpha^2}{h^2}(c^2-1)+\frac{3m^2}{h^2}\left\{1+2\alpha\frac{c}{h}(c^2-1)-\frac{8}{3}\alpha\frac{c}{h}\right\}\right.$$

$$\left.-\frac{6m\alpha^2}{h^4}-\frac{3}{2}\frac{m^2\alpha^2e^2}{h^4}\right]. \tag{16}$$

If the test-particle possesses spin angular momentum S parallel to the angular momentum of the central rotating body, then following Corinaldesi & Papapetrou (1951) and Das (1957), we obtain for small S from (16)

$$\delta\omega = 2\pi\left[\frac{3}{2}\frac{\alpha^2}{h^2}(c^2-1)+\frac{3m^2}{h^2}\left\{1+2\alpha\frac{c}{h}(c^2-1)-\frac{8}{3}\alpha\frac{c}{h}\right\}\right.$$

$$\left.-\frac{6m^2\alpha^2}{h^4}-\frac{3}{2}\frac{m^2\alpha^2e^2}{h^4}\right]-\frac{6\pi Sm^2}{Eh^3} \quad \ldots (17)$$

where E is the total energy of the test-particle.

Deflection of Massless Particles

For massless particles such as photons, gravitons (Synge 1960) and neutrinos, both h and c tend to infinity but c/h is finite. For this case, (13) is reduced to the form

$$\frac{d^2u}{d\phi^2}+u = 2m\alpha\frac{c^3}{h^3}+3\alpha^2\frac{c^2}{h^2}u+3m\left(1-\frac{8}{3}\alpha\frac{c}{h}\right)u^2-4\alpha^2u^3 \quad \ldots (18)$$

The solution of this equation, neglecting all terms of the right side, is

$$u = \cos\phi/R \quad \ldots (19)$$

Now, substituting from (19) on the right side of (18) and finding particular integrals for the different terms, the final solution is

$$u = 1/r = 2m\alpha\frac{c^3}{h^3}+\frac{\cos\phi}{R}+\frac{3\alpha^2c^2}{2Rh^2}\phi\sin\phi$$

$$+\frac{m}{R^2}\left(1-\frac{8}{3}\alpha\frac{c}{h}\right)(\cos^2\phi+2\sin^2\phi)$$

$$-\frac{\alpha^2}{R^3}\left\{\frac{3}{2}\phi\sin\phi+\frac{3}{8}\cos\phi-\tfrac{1}{2}\cos^2\phi\right\} \quad \ldots (20)$$

The total deflection of the track of a massless particle is then given by

$$\psi = 2\left(\frac{R}{r_{\phi\to\pi/2}}\right) = 2\left\{2m\alpha\frac{c^3}{h^3}+\frac{3}{4}\alpha^2\frac{c^2}{h^2}\pi+\frac{2m}{R}\left(1-\frac{8}{3}\alpha\frac{c}{h}\right)-\frac{3\alpha^2}{4R^2}\pi\right\} \ldots (21)$$

If, however, the massless particle possesses spin angular mementum S parallel to the angular momentum of the (gravitating) rotating body, then, following Corinaldesi & Papapetrou (1951) again, we obtain from (21)

$$\psi = 2\left\{2m\alpha\frac{c^3}{h^3}+\frac{3}{4}\alpha^2\frac{c^2}{h^2}\pi+\frac{2m}{R}\left(1-\frac{8}{3}\alpha\frac{c}{h}\right)-\frac{3\alpha^2}{4R^2}\pi\right\}\left(1-\frac{S}{2RE}\right) \quad \ldots (22)$$

where E is the energy of the particle. Since the spin angular momenta of a photon, a graviton and a neutrino are, respectively $\hbar$, $2\hbar$ and $\frac{1}{2}\hbar$, equation (22) shows that for the same values of c/h, E and $R\psi$ will be the least for a graviton and greatest for a neutrino.

But for a photon, graviton or neutrino with the spin parallel to their velocities in the $\theta = \pi/2$ plane, ψ in this plane is not affected by the spin (Corinaldesi Papapetrou).

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References

Boyer R. H. & Lindquist R. J. 1967 *J. Math Phys.* **8**, 265'
Boyer R. H. & Price T. G. 1965 *Proc. Camb. Phil. Soc.* **61**, 531.
Corinaldesi E. & Papapetrou A. 1951 *Proc. Roy. Soc.* **209**, 259.
Das A. 1957 *Prog. Theo. Phys.* **17**, 373.
Kerr R. P. 1963 *Phys. Rev. Letters* **11**, 327.
Hernandez W. C. 1968 *Phys. Rev.* **167**, 1180.
Synge J. L. 1960 *Relativity, The General Theory* 228.